Boson Mapping in Field Theory[1]

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A boson mapping of pair field operators is presented. The mapping preserves all hermiticity properties and the Poisson bracket relations between fields and momenta. The most practical application of the boson mapping is to field theories which exhibit bound states of pairs of fields. As a concrete application we consider, in the low energy limit, the Wick-Cutkosky model with equal mass for the charged fields.

I. Introduction

Boson mapping techniques play an important role in many areas of physics whenever the low lying states in energy are dominated by pair correlations of the fundamental particles. A typical example is the seniority model[1] where the dominant contribution to the ground state of a nuclear system is given by pairs of nucleons of total spin zero. In condensed matter physics a famous example is superconductivity where pairs of electrons are coupled via an electron-phonon interaction to form Cooper pairs.

A pair of creation operators of two fundamental particles can be used to represent the creation of a composite bosonic field if one can find a conju-

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gate annihilation operator which annihilates the field. A pair of annihilation operators of the fundamental particles is not sufficient. However, the boson mapping[1, 2] allows one to find just such an operator.

It is natural to ask if such a mapping can be applied directly at the level of field operators rather than annihilation and creation operators, the motivation being to map a Hamiltonian (Lagrangian) density to an effective one which depends only on pair operators. For example, if the Hamiltonian density of the gauge field of QCD can be mapped to one which depends only on composite operators representing gluon pairs of total spin and colour zero, then one would end up with a new effective Hamiltonian density which depends only on a scalar field. The simplification in structure is obvious. So, if at low energies the ground state is dominated by pair correlations it is very probable that a boson mapping not only simplifies the structure but also takes into account the physics. This is the case for QCD where perturbatively a pair of gluons of spin and colour zero is the lowest energy state. Additionally, in the quark-antiquark sector we have ample experimental evidence that the low energy physics is dominated by quark-antiquark pairs of spin and colour zero (e.g., the pion) and by di-quarks. However, up to now no explicit mapping at the level of field operators, depending on the coordinate $x = (\vec{x}, t_o)$, where t_o refers to a fixed time, has been given. All applications have been restricted to creation and annihilation operators[1, 2]. Recently, the boson mapping has been extended to pairs of coordinate operators $Q_{ai} = \frac{1}{\sqrt{2}}(b_{ai}^\dagger + b_{ai})$, where the indices i and a refer to a set quantum numbers, and their derivatives $\frac{\partial}{\partial Q_{ai}}$[3]. The main reason to consider such a mapping is that the Hamiltonian densities of field theories have a much simpler structure in terms of coordinate operators Q_{ai} which are the expansion coefficients of the fields $\Phi_a(x)$ in terms of an orthonormal set of functions $f_i(x)$, i.e., $\Phi_a(x) = \sum_i \frac{1}{\sqrt{\omega_i}} f_i(x) Q_{ai}$.

In this contribution we will show how to implement a boson mapping at the level of classical pair fields, e.g., $\left\{ \frac{1}{\sqrt{N}} \sum_a \Phi_a(x) \Phi^a(x) \right\} = q(x)$ with a denoting some quantum numbers, and discuss some problems related to it. Although the mapping is presented at the classical level it is relatively straightforward to check that it goes through in the quantum case where Poisson brackets are replaced by commutators. As a specific example, we will investigate the Wick-Cutkosky model[4] (WCM) which is known to have bound states. In the WCM we integrate out the scalar field which mediates the interaction between two charged fields. For equal masses of the latter

and in the limit where the mass of the scalar field is much greater than any relevant energy involved, we will obtain a model which, after boson mapping, will be equivalent to a scalar field theory with only one scalar neutral field. The classical equation of motion is that of an anharmonic oscillator with an attractive fourth order term in the potential. Consequently, the classical system is unstable as $q \to \infty$ when all other classical degrees of freedoms are neglected. It is important to emphasize that in the full quantum theory this unphysical feature is expected to disappear.

We will not discuss here the construction of the ground state and excited states of the WCM. This will be done in another publication where we will compare results obtained from the boson mapping with those derived from a more fundamental renormalization group analysis of the model and investigate the limit of applicability of the boson mapping. We will also restrict our attention here to boson fields as fundamental fields, the extension to fermion fields being straightforward.

II. A Boson Mapping for Fields:

Let us denote a general bosonic field by $\Phi_{ia}(x)$ with $i = 1, ..., M$ and $a = 1, ..., N$. The conjugate momenta to these fields are denoted by $\Pi^{ia}(x)$ and satisfy with $\Phi_{ia}(x)$ the Poisson bracket relation $\{\Phi_{jb}(\vec{y}, t_o), \Pi^{ia}(\vec{x}, t_o)\} = \delta^{ia}_{jb}\delta(\vec{x} - \vec{y})$.[5] We use co- and contravariant indices in order to allow non-cartesian components. Associated to these fields and their conjugate momenta we introduce pair fields $(q_{ij}(x))$ and their conjugate momenta $(p^{ij}(x))$ satisfying the Poisson bracket relation $\{q_{nm}(\vec{y}, t_o), p^{ij}(\vec{x}, t_o)\} = (\delta^{ij}_{nm} + \delta^{ji}_{nm})\delta(\vec{x} - \vec{y})$, where instead of the usual definition of the Poisson bracket we multiplied for convenience the right hand side by an extra factor of one half. (This comes first by noting that $\frac{\delta q_{nm}(x)}{\delta q_{ij}(y)} = (\delta^{ij}_{nm} + \delta^{ji}_{nm})\delta(x - y)$, which reflects the non-normalized nature of $q_{ij}(x)$, and that we require the above mentioned Poisson bracket relation between $q_{nm}(y)$ and $p^{ij}(x)$ be satisfied in analogy to the usual boson pair relation[2].) The $q_{ij}(x)$ is called a pair field because it will be related to a pair of the fundamental fields $\Phi_{ia}(x)$. It is still not normalized, however, as can be seen from the Poisson bracket when $(nm) = (ij)$.

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[5]The form of the Poisson bracket is given in the chapter about continous fields of the book written by H.Goldstein[5].

This is also for convenience and can later on be corrected by an appropriate normalization of the fields. Furthermore, the pair fields and their conjugate momenta are symmetric with repect to interchange of their discrete indices ($q_{ij}(x) = q_{ji}(x)$, etc.).

We now propose a mapping from fundamental to composite fields such that the Poisson bracket relations and hermiticity properties are conserved. Both conditions are important: the former to ensure that the dynamics is preserved (for at least a physically relevant subspace of the space of states) and the latter to ensure the invariance of matrix elements under the mapping. The explicit mapping is

$$
\begin{aligned}
\left\{ \frac{1}{\sqrt{N}} \sum_a \Phi_{ia}(x)\Phi_j^a(x) \right\} &= q_{ij}(x) \\
\left\{ \frac{1}{\sqrt{N}} \sum_a \Pi_a^i(x)\Pi^{ja}(x) \right\} &= \frac{1}{N} \sum_{k_1 k_2} p^{ik_1}(x)q_{k_1 k_2}(x)p^{k_2 j}(x) \qquad (1) \\
\left\{ \sum_a \Phi_{ia}(x)\Pi^{ja}(x) \right\} &= \sum_k q_{ik}(x)p^{kj}(x)
\end{aligned}
$$

where the curly bracket on the left hand side denotes the mapping, i.e., the left hand side gives the operator in the original space of states while the mapped expression gives the operator in the new space of states, defined by the pair fields. In eq.2 both sides satisfy exactly the same Poisson bracket relations as can be seen by direct calculation. The hermiticity properties are obviously satisfied, i.e., the first two expressions on the left are hermitian and so they are on the right. The last is anti-hermitian and so is the mapped expression on the right. The $\frac{1}{\sqrt{N}}$ on the left hand side is the normalization of the pair and has its origin in the coupling coefficient which couples the field to a scalar with respect to the property characterized by the index $a = 1, 2, ..., N$. For example, in the case of QCD this index refers to colour and the above objects have colour zero. The mapping of eq.2 can easily be generalized to a quantized theory, the only change being that the last operator has to be written symmetrically.

Note that a kinetic energy, quadratic in the momenta of the original fields, is mapped to a kinetic energy quadratic in the pair fields but which now depends also on the field itself and not only on the conjugate momenta. This corresponds to a "mass" parameter which depends on the strength of the field. This situation appears quite often in physics. As an example see

ref.[6] which describes the moment of inertia of a nucleus as a function of deformation. It can also be given the interpretation of working in a non-cartesian coordinate system in the configuration space of the fields as will be seen in a less general setting shortly. The appearance of the composite field explicitly in its own kinetic term indicates the high degree of non-linearity of the mapping, something that makes the explicit construction of the inverse mapping quite difficult.

We have still to consider the mapping of expressions containing derivatives such as $\sum_a (\nabla \Phi_{ia}) \cdot (\nabla \Phi_j^a)$, which appear in the original Lagrangian density. The problem here is that the field derivative is a limit of differences of its value at two neighbouring points. Thus, the above product contains products of two fields at neighbouring points. The mapping in eq.2 is only constructed for products of fields at the same point (a generalization of it with different vectors x and y can also be given but it leads to non-local field theories, which we try to avoid for the moment). For the case $M = 1$, i.e. in the case where there are no external indices, the result is

$$\left\{ \frac{1}{\sqrt{N}} \sum_a (\nabla \Phi_a(x)) \cdot (\nabla \Phi^a(x)) \right\} = \frac{(\nabla q(x)) \cdot (\nabla q(x))}{4q(x)} \quad . \qquad (2)$$

(For the general case it is more involved.)

We can also see this mapping of the derivative term, and indeed the entire boson mapping itself, by considering a change of variables in the measure of the functional integral. For instance, in the case of a complex scalar field $\Phi(x)$ we change variable $\Phi(x) = \sqrt{q(x)} exp(i\theta(x))$. The volume element $d\Phi d\Phi^*$ changes to $dq d\theta$, apart from a constant. The expression $(\partial_\mu \Phi)(\partial^\mu \Phi^*)$ becomes $\frac{(\partial_\mu q)(\partial^\mu q)}{4q}$ $+q(\partial_\mu \theta)(\partial^\mu \theta)$. For the spatial part $(\nabla \Phi) \cdot (\nabla \Phi^*)$ the first term corresponds to the mapping, when restricted to pairs $q(x)$ only. For the part $\mid \partial_o \Phi \mid^2$ we have to take the expression for the classical conjugate momentum $p(x)$ (given in the next section) whereupon we will arrive at the mapping proposed in eq.2 for the square of the momenta $\Pi(x)$. Note, that in this example we have an extra contribution $q(\partial_\mu \theta)(\partial^\mu \theta)$. Generarically this type of term becomes important when the contribution of the fields not coupled to pairs start to dominate. For that case the mapping in eq.2 has to be generalized along the same line as done in the examples of nuclear physics[2] when broken pairs are introduced. The mapping in eq.2 is, therefore, restricted (projected) to states of the space of states which are dominated by

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the pairs $q(x)$. The simple change of variables, used in the above example, is generalized in eq.2.

In the future we intend to extend the mapping in eq.2 such that pairs of fields at different space-time points are considered. As mentioned above, this leads to non-local field theories, however, by using operator product expansion methods this yields local field theories. The work is still in progress.

III. The Wick-Cutkosky model (WCM):

In the WCM[4] the Langragian, with equal masses of the charged fields, is given by

$$
\begin{aligned}
\mathcal{L}(x) \;=\; & \frac{1}{2} \sum_{b=1}^{2} (\partial_\mu \Phi_b^\dagger(x))(\partial^\mu \Phi^b(x)) + \frac{1}{2}((\partial_\mu \phi(x))(\partial^\mu \phi(x)) \\
& - \frac{M^2}{2} \sum_{b=1}^{2} \Phi_b^\dagger(x) \Phi^b(x) - \frac{\mu^2}{2} \phi^2(x) \\
& - g(\sum_{b=1}^{2} \Phi_b^\dagger(x) \Phi^b(x)) \phi(x) \quad .
\end{aligned}
\tag{3}
$$

In eq.3 the charged field is described by a complex field which in terms of real and imaginary parts implies that the sum over a involves four real fields. In this case it manifestly has the structure of eq.2[6].

Using functional integration with Euclidian measure we can integrate over the real field $\phi(x)$ using gaussian integration. We obtain finally for an effective Lagrangian density

$$
\begin{aligned}
\mathcal{L}_{eff}(x) \;=\; & \frac{1}{2} \sum_{b=1}^{2} \partial_o \Phi_b^\dagger(x) \partial_o \Phi^b(x) - \frac{1}{2} \sum_{b=1}^{2} (\nabla \Phi_b^\dagger(x)) \cdot (\nabla \Phi^b(x)) + \\
& - \frac{M^2}{2} \sum_{b=1}^{2} \Phi_b^\dagger(x) \Phi^b(x)
\end{aligned}
$$

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[6]In fact the mapping in eq.2 is the same for complex fields in which one of the fields has to be taken as the complex conjugate and the value N is twice the number of complex fields.

$$+\frac{g^2}{2}(\sum_{b=1}^{2}\Phi_b^\dagger(x)\Phi^b(x))\int d^4y\Delta_F(x-y)(\sum_{c=1}^{2}\Phi_c^\dagger(y)\Phi^c(y)) \quad (4)$$

where $\Delta_F(x-y)$ is the Feynman propagator for scalar fields.

In the limit of μ^2 much greater than the momenta involved which corresponds for a given ultraviolet cutoff Λ to $(\frac{\mu}{\Lambda})^2 >> 1$, i.e. that the momenta involved are within the low energy limit, the interaction part of the effective Lagrangian density in eq.4 acquires the form

$$\frac{g^2}{2\mu^2}(\sum_{b=1}^{2}\Phi_b^\dagger(x)\Phi^b(x))^2 \quad . \quad (5)$$

We now express the complex fields $\Phi_b(x)$ in terms of their real and imaginary part, i.e., $\Phi_b(x) = \Phi_{1,b}(x) + \Phi_{2,b}(x)$ which yields four different fields $\Phi_a(x)$ with a linearized index $a = 1,...,4$ ($a = 1$: $(1,b)$; $a = 2$: $(2,b)$; etc.). Then we change to the Hamiltonian formulation because the mapping involves the conjugate momenta. This yields for the kinetic energy the form $\frac{1}{2}\sum_{a=1}^{4}\Pi_a(x)\Pi^a(x)$ and the potential part in eq.4 (including the term which involves the square of the gradient of the fields) changes its sign. With the mapping of eq.2 (with $i,j = 1$) and the normalization of the pair fields $q(x)$ by multiplying them and their conjugate momenta by $\frac{1}{\sqrt{2}}$ (denoting them afterwards with the same letters) this Hamiltonian density is mapped to

$$\begin{aligned}\mathcal{H}_{bos}(x) &= \frac{1}{\sqrt{2}}p(x)q(x)p(x) + \sqrt{2}M^2q(x) - \frac{4g^2}{\mu^2}q^2(x) \\ &+ \frac{\sqrt{2}}{4}\frac{(\nabla q(x))\cdot(\nabla q(x))}{q(x)} \quad . \quad (6)\end{aligned}$$

The kinetic energy of the Hamiltonian density contains a dependence on the field $q(x)$ and the potential has a term $\sim q(x)$ which can be interpreted as a magnetic type of interaction. From eq.6 the momentum as a function of $q(x)$ can be deduced:

$$p(x) = \frac{1}{\sqrt{2}q(x)}\frac{\partial q(x)}{\partial t} \quad (7)$$

The equation of motion is derived from the new Hamiltonian density and is of the form

$$\Box q(x) - \frac{1}{2q(x)}(\nabla^\mu q(x)) \cdot (\nabla^\mu q(x))$$

$$+2M^2 q(x) - \frac{8\sqrt{2}g^2}{\mu^2}q^2(x) = 0 \quad . \tag{8}$$

The second term is a connection term associated with, as mentioned previously, the fact that we are using non-cartesian coordinates. Making the substitution $q(x) = \chi^2(x)$ we arrive at the equation

$$\Box\chi(x) + M^2\chi(x) - \frac{4\sqrt{2}g^2}{\mu^2}\chi^3(x) = 0 \quad . \tag{9}$$

This is the classical equation of motion of an anharmonic oscillator where the anharmonic term in the potential is attractive. This term destabilizes the system for large coupling constant g and produces a transition to a state with infinite expectation value $< q >$. It, therefore, indicates that the system is unstable under the formation of a q-condensate. Higher, repulsive terms, not included in the Hamiltonian density, should make it stable.

The approximation we have made in the above is restricting the space of states to functionals which depend on the pair fields only. It remains to be seen that at low energy this assumption is justified, i.e., that the lowest lying states are dominated by bound states which are comprised of a coupling of two fields. This is under investigation[7].

IV. Conclusion

In this contribution we proposed a mapping of fundamental pair fields and their conjugate momenta to new composite fields and their corresponding conjugate momenta. We restricted attention to the case of classical fields, with the proviso that the mapping can be directly extended to a quantized picture without undue difficulty. The mapping will be particularly useful whenever pair correlations dominate the low energy structure, as is the case for QCD. Only boson fields as fundamental fields were considered, the extension to fermionic fields being direct. (There is some relation of the boson

mapping, as presented here on the operator level, to the formulation with Feynman path integrals[8]. We thank F.J.W.Hahne for pointing this out during the conference.)

The mapping was applied to the Wick-Cutkosky model, where in the limit of low energy and equal masses for the charged fields we obtained a field theory equivalent to that of a neutral self-interacting scalar field. The classical equation of motion was that of an anharmonic oscillator with an attractive anharmonic term of fourth order in the potential. As a result, the system is unstable to small fluctuations.

What still has to be shown in a fundamental (basic) calculation is that at low energies the Wick-Cutkosky model exhibits the pair structure used. Also under investigation is how far the boson model gives correct results compared to the exact description. This is important in order to probe the range of application of the boson model and deduce from there possible applications to QCD.

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